GAUSSIAN PROCESSES
EXERCISE SHEET 7: LINEAR STRUCTURE OF GAUSSIANS

Exercise 1 (Standard Gaussians in different geometries)**.** *Let $\bar{X}$ be a Gaussian on $\mathbb{R}^n$ with covariance kernel C. Prove that $\langle x, y\rangle_C := \langle x, C^{-1}y\rangle$ defines an inner product on $\mathbb{R}^n$. Prove that $\bar{X}$ is the standard Gaussian on $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_C)$. What are its symmetries?*

Exercise 2 (Discrete White Noise)**.** *Consider the Gaussian vector $\bar{X} = (X_1, \dots, X_n)$ as a random height function $W : \{1, \dots, n\} \to \mathbb{R}$ defined by $W(n) := X_n$. In case we take $\bar{X}$ to be the standard Gaussian, this height function is called the discrete white noise. By considering functions $\phi_i(j) := \sin(\pi\frac{ji}{n+1})$ for $i = 1 \dots n$, find the discrete Fourier expansion of W.*

Exercise 3 (Gaussian random walk)**.** *Define now a Gaussian height function $\Gamma : \{0, 1, \dots, n\} \to \mathbb{R}$ by $\Gamma(0) := 0$, and $\Gamma(j) = \sum_{i=1}^{j} W(i)$. This is called the Gaussian random walk.*

Show that by choosing appropriately $c_1, \dots, c_n$, one can write $\Gamma(j) = jc_nZ_n + \sum_{i=1}^{n-1} c_iZ_i\phi_i(j)$, with $Z_1, \dots, Z_n$ being independent standard Gaussians and $\phi_i(j) := \sin(\pi\frac{ji}{n})$.

Exercise 4 (Standard Gaussian in a Hilbert space)**.** *Let H be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Fix an orthonormal basis $(e_k)_{k\geq 1}$ of H, and let $(\xi_k)_{k\geq 1}$ be i.i.d. standard normal random variables. If we roughly write a* standard Gaussian random variable X *in H as*

$$X = \sum_{k=1}^{\infty} \xi_k \, e_k,$$

so that

$$\langle X, h\rangle = \sum_{k=1}^{\infty} \xi_k \, \langle e_k, h\rangle, \qquad h \in H.$$

Show that, for any finite collection of vectors $h_1, \dots, h_n \in H$, the covariance matrix of the Gaussian vector $(\langle X, h_1\rangle, \dots, \langle X, h_n\rangle)$ is independent of the choice of orthonormal basis used to define X.